MATH 2050C Mathematical Analysis I
2019-20 Term 2
Problem Set 8

due on Mar 27, 2020 (Friday) at 11:59PM

Instructions: You are allowed to discuss with your classmates or seek help from the TAs but you are required to write/type up your own solutions. please do NOT come to campus to submit your completed assignments. Instead, you can either type up your assignment or scan a copy of your written assignment into ONE PDF file and submit through CUHK Blackboard on/before the due date. Please remember to write down your name and student ID. You can refer to the webpage under "Useful Links" below about how to submit assignments through Blackboard. **No late homework will be accepted.** All the exercises below are taken from the textbook.

Required Readings: Chapter 3.5 and 4.1

Optional Readings: Chapter 3.6 and 3.7

Problems to hand in

Section 3.5: Exercise # 2(a), 3(c), 7, 10

Section 4.1: Exercise # 8, 9(d), 12(d), 13

Suggested Exercises

Section 3.5: Exercise # 1, 2(b), 3(a)(b), 4, 5, 8, 9, 11,12, 13

Section 4.1: Exercise # 3, 4, 5, 6, 7, 9, 10, 11, 12, 14, 15

Challenging Exercises (optional)

1. Section 3.5: Exercise # 6
2. Recall that a subset $A \subset \mathbb{R}$ is *dense* if $A \cap (a, b) \neq \emptyset$ for any $a < b$. A subset $B \subset \mathbb{R}$ is said to be *open* if for any $b \in B$, there exists some $\epsilon > 0$ such that $(b - \epsilon, b + \epsilon) \subset B$. Prove *Baire's Theorem*: if $\{I_n\}_{n\in\mathbb{N}}$ is a sequence of dense open subsets in $\mathbb{R}$, then $\cap_{n=1}^{\infty} I_n$ is non-empty (in fact dense in $\mathbb{R}$).

3. Let (x_n) and (y_n) be Cauchy sequences in $\mathbb{R}$. Define a new sequence $z_n := |x_n - y_n|$, $n \in \mathbb{N}$. Prove that (z_n) is convergent.

4. This exercises show you how to construct the real numbers $\mathbb{R}$ from rational numbers $\mathbb{Q}$ using the concept of Cauchy sequences. We say that a sequence (q_n) of rational numbers is Cauchy if for any $\epsilon > 0$, there exists $H = H(\epsilon) \in \mathbb{N}$ such that $|q_n - q_m| < \epsilon$ for all $m, n \geq H$.

   (a) Two Cauchy sequences (p_n) and (q_n) of rational numbers are said to be *equivalent*, written $(p_n) \sim (q_n)$, if

   $$\lim(|p_n - q_n|) = 0.$$

   Prove that $\sim$ defines an equivalence relation on the set of all Cauchy sequences in $\mathbb{Q}$.

   (b) Let $\mathbb{Q}^*$ be the set of all equivalence classes of Cauchy sequences in $\mathbb{Q}$ as defined in (a). If $P, Q \in \mathbb{Q}^*$ and $(p_n) \in P$, $(q_n) \in Q$, we define

   $$d(P, Q) := \lim(|p_n - q_n|).$$

   Show that d is well-defined (i.e. independent on the choices of representatives (p_n) and (q_n) in their equivalence classes) and gives a distance function on $\mathbb{Q}^*$ satisfying:

   (i) $d(P, Q) \geq 0$ and equality holds only if $P = Q$

   (ii) $d(P, Q) = d(Q, P)$

   (iii) $d(P, Q) \leq d(P, R) + d(R, Q)$

   (c) Prove that $\mathbb{Q}^*$ is *complete* with respect to d, i.e. every Cauchy sequences in $\mathbb{Q}^*$ converges to a limit in $\mathbb{Q}^*$ (all with respect to the distance d).

   (d) For each $p \in \mathbb{Q}$, denote $\overline{p} := (p, p, p, \cdots)$ as the constant sequence. Show that for any $p, q \in \mathbb{Q}$,

   $$d([\overline{p}], [\overline{q}]) = |p - q|.$$

   Thus, the mapping $\iota : \mathbb{Q} \to \mathbb{Q}^*$ given by $\iota(p) = [\overline{p}]$ is an isometry.

   (e) Prove that $\iota(\mathbb{Q})$ is dense in $\mathbb{Q}^*$.

   (f) Identify $\mathbb{Q}^*$ with $\mathbb{R}$.

5. Section 4.1: Exercise # 16, 17